A notion of multivariate Value at Risk from a directional perspective

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OUTLINE

1. INTRODUCTION
2. DIRECTIONAL MULTIVARIATE VALUE AT RISK (MVAR)
3. MARGINAL VAR VS. MVAR
4. COPULAS AND $VaR_{\alpha}^{\mathbf{u}}(\mathbf{X})$
5. NON-PARAMETRIC ESTIMATION
6. ROBUSTNESS
7. CONCLUSIONS

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VALUE AT RISK (*VaR*)

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$$VaR_\alpha(X) := \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\}.$$

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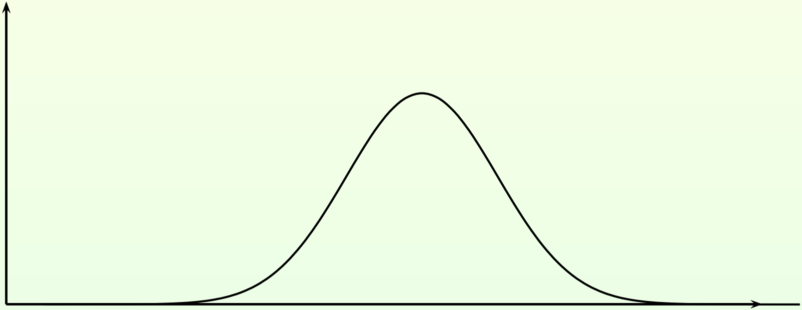

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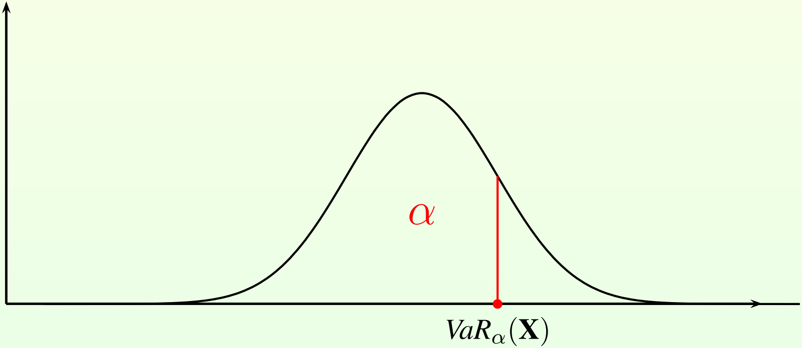

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- The *VaR* has became in a benchmark for risk management.
- The *VaR* has been criticized by Artzner et al. (1999) since it does not encourage diversification.
- But defended by Heyde et al. (2009) for its robustness and recently by Daníelsson et al. (2013) for its tail subadditivity.

Value at Risk (*VaR*)

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It is its extension to the multivariate setting

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But, what is one of the problems with this measure?

It is its extension to the multivariate setting, where

- **There is not a unique definition of a multivariate quantile.**
- **There are a lot of assets in a portfolio. (High Dimension)**
- **There is dependence among them.**

Review on Multivariate Value at Risk

An initial idea to study risk measures related to portfolios

$$\mathbf{X} = (X_1, \ldots, X_n),$$

is to consider a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ **and then:**

- **The *VaR* of the joint portfolio is the univariate-one associated to** $f(\mathbf{X})$**.**

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is to consider a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and then:

- **The *VaR* of the joint portfolio is the univariate-one associated to $f(\mathbf{X})$.**
- **In Burgert and Rüschendorf (2006),**

$$f(\mathbf{X}) = \sum_{i=1}^{n} X_i \quad \text{or} \quad f(\mathbf{X}) = \max_{i \le n} X_i.$$

Output: A NUMBER

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- **Multivariate lower-orthant Value at Risk**

$$\underline{VaR}_\alpha(\mathbf{X}) := \partial\{\mathbf{x} \in \mathbb{R}^n \mid F_{\mathbf{X}}(\mathbf{x}) \geq \alpha\}.$$

- **Multivariate upper-orthant Value at Risk**

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Output: A SURFACE ON $\mathbb{R}^n$

Review on Multivariate Value at Risk

Cousin and Di Bernardino (2013) introduced a multivariate risk measure related to the measure introduced by Embrechts and Puccetti (2006).

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Output: A POINT IN $\mathbb{R}^n$

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- **The computation in high dimensions.**

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Introduce a directional multivariate value at risk

1. **Consider the dependence among the variables.**
2. **Give the possibility of analyzing the losses considering the manager preferences.**
3. **Improve the interpretation of the risk measure.**
4. **Provide a non-parametric estimation to compute the risk measure in high dimensions.**
5. **Provide analytic expressions of the risk measure with copulas.**

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Directional Multivariate Value at Risk (*MVaR*)

Directional MVaR

Let $\mathbf{X}$ be a random vector satisfying "*the regularity conditions*", then the Value at Risk of $\mathbf{X}$ in direction $\mathbf{u}$ and *confidence parameter* α is defined as

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) = \left(\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) \bigcap \{\lambda \mathbf{u} + \mathbb{E}[\mathbf{X}]\} \right),$$

where $\lambda \in \mathbb{R}$ and $0 \leq \alpha \leq 1$.

Output: A POINT IN $\mathbb{R}^n$

$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) \quad \equiv \quad$ **Directional Multivariate Quantile (Laniado et al. (2012)).**

DEFINITION

Given $\mathbf{u} \in \mathbb{R}^n$, $||\mathbf{u}|| = 1$ and a random vector $\mathbf{X}$ with distribution probability $\mathbb{P}$, the α-quantile curve in direction $\mathbf{u}$ is defined as:

$$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) := \partial\{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}\left[\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}}\right] \leq \alpha\},$$

where ∂ mans the boundary and $0 \leq \alpha \leq 1$

$\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} \equiv$ Oriented Orthant.

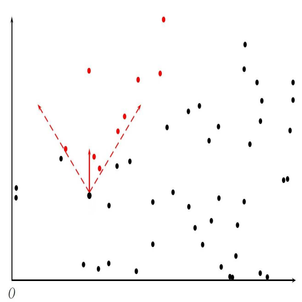
Definition

Given $\mathbf{x}$, $\mathbf{u} \in \mathbb{R}^n$ and $||\mathbf{u}|| = 1$, the orthant with vertex $\mathbf{x}$ and direction $\mathbf{u}$ is:

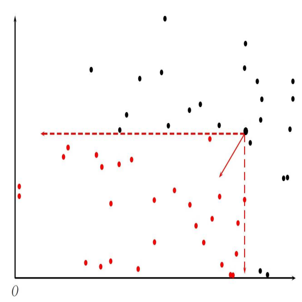
$$\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} = \{\mathbf{z} \in \mathbb{R}^n | R_{\mathbf{u}}(\mathbf{z} - \mathbf{x}) \geq 0\},$$

where $\mathbf{e} = \frac{1}{\sqrt{n}}(1, ..., 1)'$ and $R_{\mathbf{u}}$ is a matrix such that $R_{\mathbf{u}}\mathbf{u} = \mathbf{e}$.

Examples of Oriented Orthants



(A) Orthant in direction $\mathbf{u} = (0, 1)$

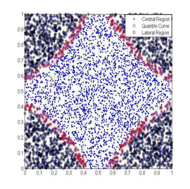


(B) Orthant in direction $\mathbf{u} = -\mathbf{e}$

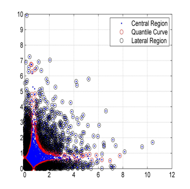
Examples of oriented orthants in $\mathbb{R}^2$

Directional Multivariate Quantiles

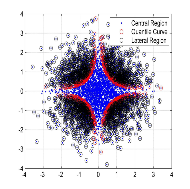
$$\mathbf{u} \in \mathfrak{U} = \left\{ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$$



(A) Bivariate Uniform



(B) Bivariate Exponential

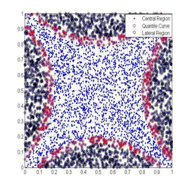


(C) Bivariate Normal

CLASSICAL DIRECTIONS

Directional Multivariate Quantiles

$$\mathbf{u} \in \mathfrak{U} = \{(1,0), (0,1), (-1,0), (0,-1)\}$$



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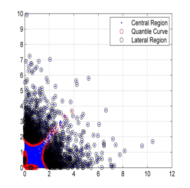
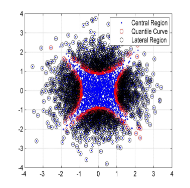


(B) Bivariate Exponential



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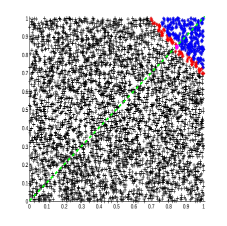
CANONICAL DIRECTIONS

Directional Multivariate Value at Risk (*MVaR*)



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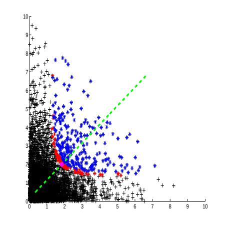
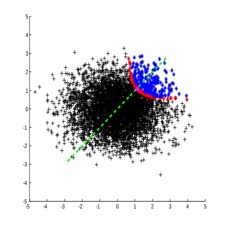


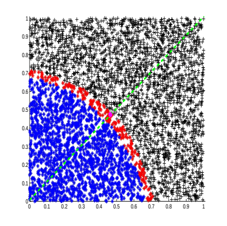
(C) Bivariate Exponential



(B) Bivariate Normal

$$VaR_{0.7}^{-\mathbf{e}}(\mathbf{X})$$

DIRECTIONAL MULTIVARIATE VALUE AT RISK (*MVaR*)



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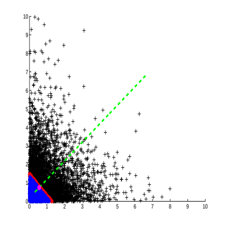

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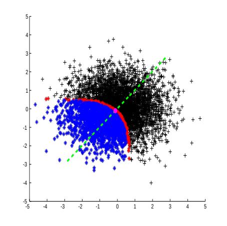


(B) Bivariate Normal

$$VaR^{\mathbf{e}}_{0.3}(\mathbf{X})$$

MVaR Properties

- **Non-Negative Loading**: If $\lambda > 0$,

$$\mathbb{E}[\mathbf{X}] \preceq_{\mathbf{u}} VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}),$$

where the order is given by

Preorder (Laniado et al. (2010))

$\mathbf{x}$ is said to be less than $\mathbf{y}$ if:

$$\mathbf{x} \preceq_{\mathbf{u}} \mathbf{y} \quad \equiv \quad \mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} \supseteq \mathfrak{C}_{\mathbf{y}}^{\mathbf{u}} \quad \equiv \quad R_{\mathbf{u}}\mathbf{x} \leq R_{\mathbf{u}}\mathbf{y}.$$

MVaR Properties

- **Quasi-Odd Measure**: $VaR_\alpha^{\mathbf{u}}(-\mathbf{X}) = -VaR_\alpha^{-\mathbf{u}}(\mathbf{X})$.

- **Positive Homogeneity and Translation Invariance**: Given $c \in \mathbb{R}^+$ and $\mathbf{b} \in \mathbb{R}^n$, then

$$VaR_\alpha^{\mathbf{u}}(c\mathbf{X} + \mathbf{b}) = cVaR_\alpha^{\mathbf{u}}(\mathbf{X}) + \mathbf{b}.$$

MVaR Properties

- **Orthogonal Quasi-Invariance**: Let $\mathbf{w}$ and Q be an unit vector and a particular orthogonal matrix obtained by a QR decomposition such that $Q\mathbf{u} = \mathbf{w}$. Then,

$$VaR_{\alpha}^{\mathbf{w}}(Q\mathbf{X}) = QVaR_{\alpha}^{\mathbf{u}}(\mathbf{X}).$$

MVaR Properties

- **Consistency**: Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors such that $\mathbb{E}[\mathbf{Y}] = c\mathbf{u} + \mathbb{E}[\mathbf{X}]$, for $c > 0$ and $\mathbf{X} \leq_{\mathcal{E}_\mathbf{u}} \mathbf{Y}$. Then:

$$VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}) \preceq_{\mathbf{u}} VaR^{\mathbf{u}}_{\alpha}(\mathbf{Y}),$$

where the stochastic order is defined by

Stochastic Extremality Order (Laniado et al. (2012))

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors in $\mathbb{R}^n$,

$$\mathbf{X} \leq_{\mathcal{E}_\mathbf{u}} \mathbf{Y} \quad \equiv \quad \mathbb{P}\left[R_\mathbf{u}(\mathbf{X}-\mathbf{z}) \geq 0\right] \leq \mathbb{P}\left[R_\mathbf{u}(\mathbf{Y}-\mathbf{z}) \geq 0\right] \quad \equiv \quad \mathbb{P}_\mathbf{X}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{z}}\right] \leq \mathbb{P}_\mathbf{Y}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{z}}\right],$$

for all $\mathbf{z}$ in $\mathbb{R}^n$.

MVaR Properties

- **Non-Excessive Loading**: For all $\alpha \in (0,1)$ and $\mathbf{u} \in \mathbb{B}(0)$,

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) \preceq_{\mathbf{u}} R'_{\mathbf{u}} \sup_{\omega \in \Omega} \{R_{\mathbf{u}} \mathbf{X}(\omega)\}.$$

- **Subadditivity in the Tail Region**: Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors, with the same mean μ and let $(R_{\mathbf{u}}\mathbf{X}, R_{\mathbf{u}}\mathbf{Y})$ be a regularly varying random vector. Then,

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}+\mathbf{Y}) \preceq_{\mathbf{u}} VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) + VaR_{\alpha}^{\mathbf{u}}(\mathbf{Y}).$$

LOWER AND UPPER VERSIONS OF DIRECTIONAL *MVaR*

RESULT

Let $\mathbf{X}$ be a random vector and $\mathbf{u}$ a direction. Then for all $0 \leq \alpha \leq 1$,

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) \preceq_{\mathbf{u}} VaR_{1-\alpha}^{-\mathbf{u}}(\mathbf{X}).$$

LOWER AND UPPER VERSIONS OF DIRECTIONAL *MVaR*

Then, analogously as Embrechts and Puccetti (2006) and Cousin and Di Bernardino (2013), we can define:

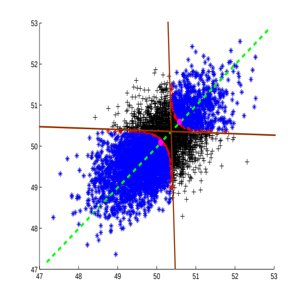
***Lower Multivariate VaR* in the direction $\mathbf{u}$ as**

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}),$$

***Upper Multivariate VaR* in the direction $\mathbf{u}$ as**

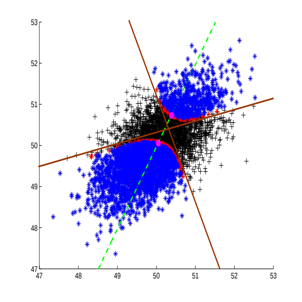
$$VaR_{1-\alpha}^{-\mathbf{u}}(\mathbf{X}).$$

LOWER AND UPPER VERSIONS OF DIRECTIONAL *MVaR*



Lower Multivariate VaR $= VaR^{\mathbf{e}}_{0.3}(\mathbf{X})$ and
Upper Multivariate VaR $= VaR^{-\mathbf{e}}_{0.7}(\mathbf{X})$

LOWER AND UPPER VERSIONS OF DIRECTIONAL *MVaR*



$$\textit{Lower Multivariate VaR} = VaR_{0.3}^{(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})}(\mathbf{X}) \text{ and}$$

$$\textit{Upper Multivariate VaR} = VaR_{0.7}^{-(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})}(\mathbf{X})$$

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RELATION BETWEEN THE MARGINAL *VaR* AND THE *MVaR*

RESULT

Let $\mathbf{X}$ be a random vector with survival function $\bar{F}$ quasi-concave. Then, for all $\alpha \in (0,1)$:

$$VaR_{1-\alpha}(X_i) \geq \left[VaR_{\alpha}^{\mathbf{e}}(\mathbf{X})\right]_i, \qquad \text{for all} \quad i = 1, ..., n.$$

Moreover, if its distribution function F is quasi-concave, then, for all $\alpha \in (0,1)$,

$$\left[VaR_{1-\alpha}^{-\mathbf{e}}(\mathbf{X})\right]_i \geq VaR_{1-\alpha}(X_i), \qquad \text{for all} \quad i = 1, ..., n.$$

Relation between the marginal *VaR* and the *MVaR*

Result

Let $\mathbf{X}$ be a random vector and $\mathbf{u}$ a direction. If the survival function of $R_{\mathbf{u}}\mathbf{X}$ is quasi-concave. Then, for all $0 \leq \alpha \leq 1$,

$$VaR_{1-\alpha}([R_{\mathbf{u}}\mathbf{X}]_i) \geq [R_{\mathbf{u}} VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})]_i\,, \qquad \text{for all} \quad i = 1, ..., n.$$

And if $R_{\mathbf{u}}X$ has a quasi-concavity cumulative distribution, we have that

$$\left[R_{\mathbf{u}} VaR^{-\mathbf{u}}_{1-\alpha}(\mathbf{X})\right]_i \geq VaR_{1-\alpha}([R_{\mathbf{u}}X]_i), \qquad \text{for all} \quad i = 1, ..., n.$$

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Bivariate Copulas

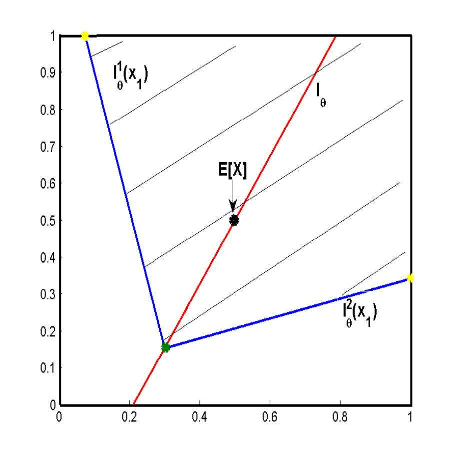
- In $\mathbb{R}^2 \Rightarrow \mathbf{u} = (\cos\theta, \sin\theta)$.
- Let $\mathbf{X}$ be a bivariate vector with density given by a copula density $c(\cdot,\cdot)$. Then, the first component of $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ can be obtained by solving the equation on the domain,

$$\int\int_{D_\theta(x_1)} c(s,t)dtds = \alpha,$$

where $D_\theta(x_1) = \mathfrak{C}^{\mathbf{u}}_{(x_1, l_\theta(x_1))} \bigcap [0,1]^2$ and

$$l_\theta(x_1) := \begin{cases} \frac{x_1 \sin(\theta) - \frac{1}{2}(\sin(\theta) - \cos(\theta))}{\cos(\theta)}, & \text{if} \quad \cos(\theta) \neq 0 \text{ and } x_1 \in [0,1], \\ \frac{1}{2}, & \text{if} \quad \cos(\theta) = 0 \text{ and } x_1 \in [0,1]. \end{cases}$$

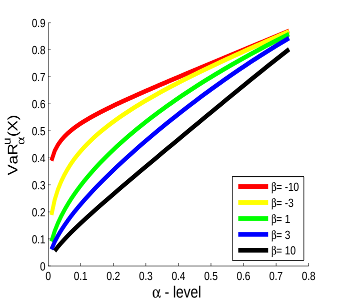
Bivariate Copulas

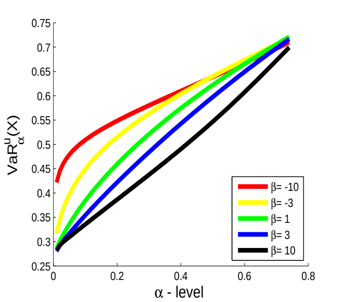


Example of $D_\theta(x_1)$ for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$

BIVARIATE COPULAS

Results of $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ with the Frank's copula, for different values on the dependence parameter β:





a) Direction $\mathbf{u} = -\mathbf{e}$

b) Direction $\mathbf{u} = -\frac{3\sqrt{5}}{5}[\frac{1}{3}, \frac{2}{3}]'$

Behavior of the first component of $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$

n-dimensional Archimedean Copulas

Let $\mathbf{X}$ be a n-dimensional random vector with $[0,1]$-uniform marginals.

- If $\mathbf{X}$ has an Archimedean copula distribution generated by $\phi(\cdot)$, then:

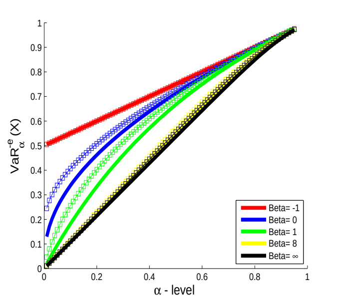
$$\left[VaR^{-\mathbf{e}}_{1-\alpha}(\mathbf{X})\right]_i = \phi^{-1}\left(\frac{1-\phi(\alpha)}{n}\right).$$

- If $\mathbf{X}$ has a survival copula given by an Archimedean copula generated by $\bar{\phi}(\cdot)$, then:

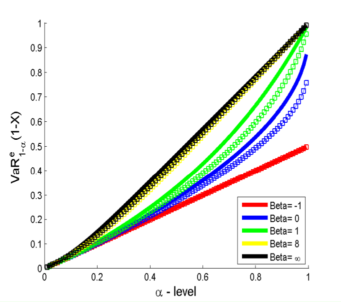
$$[VaR^{\mathbf{e}}_{\alpha}(\mathbf{X})]_i = 1 - \bar{\phi}^{-1}\left(\frac{\bar{\phi}(\alpha)}{n}\right).$$

n-DIMENSIONAL ARCHIMEDEAN COPULAS

Then, we compare $VaR^{-\mathbf{e}}_{\alpha}(\mathbf{X})$ (Our) with $\underline{VaR}_{\alpha}(\mathbf{X})$ (Cousin and Di Bernardino (2013)) and $VaR^{\mathbf{e}}_{1-\alpha}(\mathbf{1}-\mathbf{X})$ with $\overline{VaR}_{\alpha}(\mathbf{1}-\mathbf{X})$, using the Clayton's family of copulas.



a) Lower Case



b) Upper Case

Dashed line ≡ Cousin and Di Bernardino. Solid line ≡ $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$

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NON-PARAMETRIC ESTIMATION

Given the sample $\mathbf{X}_m := \{\mathbf{x_1}, \cdots, \mathbf{x_m}\}$ of the random loss $\mathbf{X}$, the direction $\mathbf{u}$ and the value of α. We find the directional quantile curve as:

$$\hat{\mathcal{Q}}_{\mathbf{X}_m}(\alpha, \mathbf{u}) := \left\{\mathbf{x_i} : \mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x_i}}\right] = \alpha\right\},$$

where

$$\mathbb{P}_{\mathbf{X}_m}[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x_i}}] = \frac{1}{m}\sum_{j=1}^{m} \mathbb{1}_{\{\mathbf{x_j} \in \mathfrak{C}^{\mathbf{u}}_{\mathbf{x_i}}\}}.$$

Non-Parametric Estimation

However, it is possible that $\hat{\mathcal{Q}}_{\mathbf{X}_m}(\alpha, \mathbf{u}) = \emptyset$. This can be solved allowing a slack h:

$$\hat{\mathcal{Q}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u}) := \left\{ \mathbf{x_j} : |\mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x_j}}\right] - \alpha| \leq h \right\},$$

where $\hat{\mathcal{Q}}_{\mathbf{X}_m}(\alpha, \mathbf{u}) \subset \hat{\mathcal{Q}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u})$, for all h.

Once the directional α-quantile curve is obtained, we cross it with the line $\{\boldsymbol{\mu}_{\mathbf{X}_m} + \lambda\mathbf{u}\}$ where

$$\boldsymbol{\mu}_{\mathbf{X}_m} = \mathbb{E}[\mathbf{X}_m].$$

Non-Parametric Estimation

Input: $\mathbf{u}$, α, h and the multivariate sample $\mathbf{X}_m$.

for $i = 1$ to m

$P_i = \mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x}_i}\right]$,

If $|P_i - \alpha| \leq h$

$\mathbf{x}_i \in \hat{\mathcal{Q}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u})$,

end

for $\mathbf{x}_j \in \hat{\mathcal{Q}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u})$

$d_j = dist(\mathbf{x}_j, \{\boldsymbol{\mu}_{\mathbf{X}_m} + \lambda\mathbf{u}\})$,

end

end

$VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}_m) = \{\mathbf{x}_k | d_k = \min\{d_j\}\}$.

Execution Time

Time in Seconds

Dim\ Size	1000	5000	10000	50000
5	2	49	199	4903
10	2	53	208	5191
50	4	82	325	7656
100	6	139	561	12487

In an Intel core i7 (3,4 GH) computer with 32 Gb RAM.

Outline

1. Introduction
2. Directional Multivariate Value at Risk (MVaR)
3. Marginal VaR vs. MVaR
4. Copulas and $VaR_\alpha^{\mathbf{u}}(\mathbf{X})$
5. Non-Parametric Estimation
6. **Robustness**
7. Conclusions

Robustness

We analyze the behavior of the *MVaR* when a sample is contaminated with different types of outliers.

We use as a benchmark the measurement given by the multivariate *VaR* in Cousin and Di Bernardino (2013).

Robustness

We simulate 5000 observations of the following random vector:

$$\mathbf{X}^{\omega} \overset{\mathbf{d}}{=} \begin{cases} \mathbf{X}_1 & \textbf{with probability } p = 1-\omega, \\ \mathbf{X}_2 & \textbf{with probability } p = \omega, \end{cases}$$

where $\mathbf{X}_1 \overset{\mathbf{d}}{=} N_1(\boldsymbol{\mu}_1, \Sigma_1)$**,** $\mathbf{X}_2 \overset{\mathbf{d}}{=} N_2(\boldsymbol{\mu}_1 + \Delta_{\boldsymbol{\mu}}, \Sigma_1 + \Delta_{\boldsymbol{\Sigma}})$ **and** $0 \leq \omega \leq 1$**. Specifically:**

$$\boldsymbol{\mu}_1 = [50, 50]', \qquad \Sigma_1 = \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{pmatrix}.$$

Contaminating
1. **Varying only the mean.**
2. **Varying only the variances.**
3. **Varying all the parameters.**

ROBUSTNESS

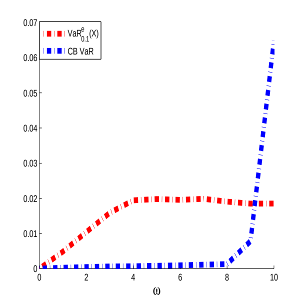
To evaluate the impact of the contamination, we use:

$$PV^{\omega} = \frac{||Measure(\mathbf{X}^{\omega}) - Measure(\mathbf{X}^{0})||_2}{||Measure(\mathbf{X}^{0})||_2},$$

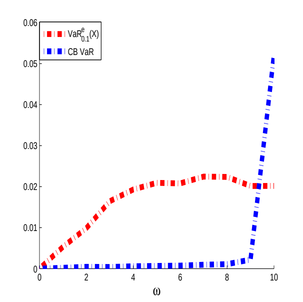
where $Measure(\mathbf{X}^0)$ **is the sample with** $\omega = 0\%$ **and** $Measure(\mathbf{X}^{\omega})$ **is the sample with level of contamination** $\omega\%$, **(**$\omega = 1\% \to 10\%$**).**

ROBUSTNESS

1. Varying only the mean, $\Delta_{\mu} \neq 0, \quad \Delta_{\Sigma} = 0$.



(A) $\Delta_{\mu} = (20, 20)'$

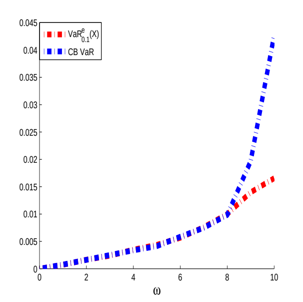


(B) $\Delta_{\mu} = (0, 50)'$

Mean of PV^{ω}

Robustness

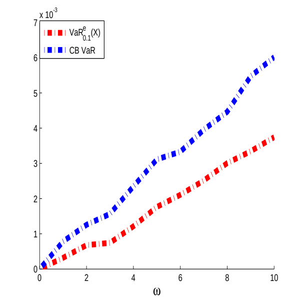
2. Varying only the variances, $\Delta_{\mu} = 0, \quad \Delta_{\Sigma} = \begin{bmatrix} 4.5 & 0 \\ 0 & 6.5 \end{bmatrix},$



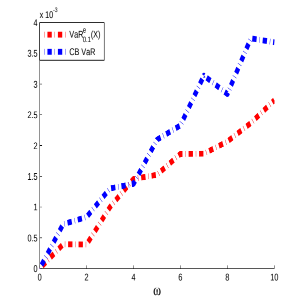
Mean of PV^{ω}

Robustness

3. Varying all the parameters, $\Delta_{\boldsymbol{\mu}} \neq 0, \quad \Delta_{\boldsymbol{\Sigma}} = \begin{bmatrix} 4.5 & 0.2 \\ 0.3 & 6.5 \end{bmatrix},$



(A) $\Delta_{\boldsymbol{\mu}} = (20, 20)'$



(B) $\Delta_{\boldsymbol{\mu}} = (0, 50)'$

Mean of PV^{ω}

OUTLINE

1. INTRODUCTION
2. DIRECTIONAL MULTIVARIATE VALUE AT RISK (MVAR)
3. MARGINAL VAR VS. MVAR
4. COPULAS AND $VaR_{\alpha}^{\mathbf{u}}(\mathbf{X})$
5. NON-PARAMETRIC ESTIMATION
6. ROBUSTNESS
7. CONCLUSIONS

Conclusions

- **We introduce a directional multivariate value at risk and a non-parametric estimation for this risk measure.**

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- We provide good properties for this risk measure, including the tail subadditivity property.

CONCLUSIONS

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- The directional approach allows to consider external information or management preferences in the analysis of the data.
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- We obtain analytic expressions with copulas.

CONCLUSIONS

- We introduce a directional multivariate value at risk and a non-parametric estimation for this risk measure.
- The directional approach allows to consider external information or management preferences in the analysis of the data.
- We provide good properties for this risk measure, including the tail subadditivity property.
- We obtain analytic expressions with copulas.
- The simulation study of robustness shows good behavior of the measure.

Thanks

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Thanks