Solution to the heat equation via symmetry analysis

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Objetive of the presentation

Through of the symmetry analysis, a solution is given to the heat equation

$$\Delta_{g^F} u = u_t \tag{1}$$

defined in the Riemannian manifold $\mathcal{M}$ induced by the family of Gaussian distributions of parameters (μ, σ), where g^F is the Fisher metric with diagonal representation

$$g^F = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix}. \tag{2}$$

A group is a set G together with a group operation, usually called multiplication, such that for any two elements g and h of G, the product $g \cdot h$ is again an element of G. The group operation is required to satisfy the axioms of: associative, modulative and there is an inverse for each element in the set.

An r-parameter Lie group is a group G which also carries the structure of an r-dimensional smooth manifold in such a way the both the group operation $m : G \times G \to G$ defined by $m(g, h) = g \cdot h$ and the inversion $i : G \to G$ defined by $i(g) = g^{-1}$ are smooth maps between manifolds.

Symmetries

A symmetry is a transformation that leaves some particular object invariant. For example, transformations $\bar{x} = e^{\xi}x$ and $\bar{y} = e^{-\xi}y$ leave the differential equation $\frac{dy}{dx} = xy^3$ invariant since

$$\frac{d\bar{y}}{d\bar{x}} = e^{-2\xi}\frac{dy}{dx} = \left[e^{\xi}x\right]\left[e^{-3\xi}y^3\right] = \bar{x}\bar{y}^3 .$$

The transformations $\bar{x}_i = f_i(x_j, \xi)$ under the composition operation form a uniparameter Lie group, where ξ is the parameter. Some of the properties they fulfill are:

1. f_i is a smooth function of the variable x_j.
2. f_i is an analytic function in the parameter ξ.
3. $\xi = 0$ can be chosen as the identity element of the group.

Infinitesimal Transformation

For the x and y variables, these transformations are written as

$$\bar{x} = f(x, y, \xi) \qquad \text{and} \qquad \bar{y} = g(x, y, \xi)$$

where $x = f(x, y, 0)$ and $y = g(x, y, 0)$. According to the expansion of the Taylor series, the transformations $\bar{x}$ and $\bar{y}$ they can be seen as

$$\begin{aligned} \bar{x} =& f(x, y, 0) + \frac{df}{d\xi}\Big|_{\xi=0} \xi + O(\xi^2) = x + X(x, y)\xi + O\left(\xi^2\right) , \\ \bar{y} =& g(x, y, 0) + \frac{dg}{d\xi}\Big|_{\xi=0} \xi + O(\xi^2) = y + Y(x, y)\xi + O\left(\xi^2\right) . \end{aligned}$$

These equalities are called **infinitesimal transformations** and $X(x, y)$ and $Y(x, y)$ are called **infinitesimals**.

Example 1: Riccati equation

Consider the Riccati equation

$$\frac{dy}{dx} = y^2 - \frac{y}{x} - \frac{1}{x^2} \; . \tag{3}$$

whose solution depends of a known solution. Table 1 shows two transformations and the consequences they bring about this equation. The equation $\frac{ds}{dr}$ obtained is of separable variables and it is also invariant for the transformations $\bar{r} = r$ and $\bar{s} = s + \xi$.

Transformations	$x = e^s,\ y = re^{-s}$	$\bar{x} = e^{\xi}x,\ \bar{y} = e^{-\xi}y$
Consequences	$\frac{ds}{dr} = \frac{1}{r^2-1}$	$\frac{d\bar{y}}{d\bar{x}} = \bar{y}^2 - \frac{\bar{y}}{\bar{x}} - \frac{1}{\bar{x}^2}$.

Table 1: Transformations in the Riccati equation.

There is a connection between the transformations

$$x = e^{s} \ , \qquad \bar{x} = e^{\xi} x \ ,$$

$$y = r e^{-s} \ . \qquad \bar{y} = e^{-\xi} y \ .$$

Let us see how to relate the infinitesimals $X(x,y)$ y $Y(x,y)$ with the variables s and r that transform the differential equation to separable variables. Let $r = r(x,y)$ and $s = s(x,y)$, invariant equations under $\bar{x}$ and $\bar{y}$, that is, $\bar{r} = r(\bar{x},\bar{y})$ and $\bar{s} = s(\bar{x},\bar{y})$. Derivating respect to ξ you have

$$\frac{\partial \bar{r}}{\partial \xi} = \frac{\partial r}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial \xi} + \frac{\partial r}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \xi} \quad \text{and} \quad \frac{\partial \bar{s}}{\partial \xi} = \frac{\partial s}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial \xi} + \frac{\partial s}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \xi} \tag{4}$$

Since $\bar{r} = r$ and $\bar{s} = s + \xi$, and making $\xi = 0$ equalities result

$$\underbrace{X(x,y)\frac{\partial r}{\partial x} + Y(x,y)\frac{\partial r}{\partial y} = 0}_{Xr_x + Yr_y = 0} \quad \text{and} \quad \underbrace{X(x,y)\frac{\partial s}{\partial x} + Y(x,y)\frac{\partial s}{\partial y} = 1}_{Xs_x + Ys_y = 1} \;. \tag{5}$$

Whose solution determines the relationship between the infinitesimals and the variables r and s.

Riccati equation

For the Riccati (3), the infinitesimal transformations given by $\bar{x} = e^{\xi}x$ and $\bar{y} = e^{-\xi}y$. The infinitesimals associated with these are:

$$X(x,y) = \frac{d\bar{x}}{d\xi}\Big|_{\xi=0} = x \quad \text{and} \quad Y(x,y) = \frac{d\bar{y}}{d\xi}\Big|_{\xi=0} = -y\ .$$

These infinitesimals induce the equalities

$$xr_x - yr_y = 0 \quad \text{and} \quad xs_x - ys_y = 1\ . \tag{6}$$

By characteristics method, the solutions to equations (6) are $r = R(xy)$ and $s = \ln(x) + S(xy)$. If R is chosen as the identity function and S as the null function then $r = xy$ and $s = \ln(x)$ where $x = e^s$ y $y = re^{-s}$, that equalities reduce the Riccati equation in an equation of separable variables.

Below is the relationship between some of these elements for classical ordinary differential equations

Equations	Linear	Riccati
Base form	$\frac{dy}{dx} + P(x)y = Q(x)$	$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$
Infinitesimal Transformations	$\bar{x} = x$ $\bar{y} = y + \xi e^{-\int P(x)dx}$	$X = 0,\ Y = (y - y_1)F(x)$ y_1 is a solution and $F(x)$ satisfies $F' + (2Py_1 + Q)F = 0$
System of Transformations (s, r)	$x = r$ $y = e^{-\int P(r)dr}\ s$	$x = r$ $y = y_1 - \frac{1}{sF(r)}$
New Equation	$\frac{ds}{dr} = e^{\int P(r)dr} Q(r)$	$\frac{ds}{dr} = \frac{a(r)}{F(r)}$

Table 2: Results for the Linear and Riccati equations.

Given a differential equation, How do we obtain the infinitesimals $X(x, y)$ and $Y(x, y)$?

Lie Invariance Condition

We will look for invariances for the equation $\frac{dy}{dx} = F(x,y)$ under the infinitesimal transformations $\bar{x}$ and $\bar{y}$. By chain rule we have

$$\frac{d\bar{y}}{d\bar{x}} = \frac{dy}{dx} + \left[Y_x + (Y_y - X_x)\, y' - X_y (y')^2\right] \xi + O\left(\xi^2\right) \ . \tag{7}$$

We know that $\frac{d\bar{y}}{d\bar{x}} = F(\bar{x}, \bar{y})$ and by the Taylor expansion for the F function of ξ results

$$F(\bar{x}, \bar{y}) = F(x,y) + [XF_x + YF_y]\, \xi + O\left(\xi^2\right) \tag{8}$$

Between the equations (7) and (8) we have the condition of Lie invariance

$$Y_x + (Y_y - X_x)F - X_y F^2 = XF_x + YF_y \ . \tag{9}$$

However, solving this equation will not be easy for PDEs.

The infinitesimal Operator

In the invariance condition applied to differential equation $\frac{dy}{dx} = F(x, y)$, the term $XF_x + YF_y$ let us define the **infinitesimal operator**

$$\Gamma = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} \ . \tag{10}$$

The relations between the infinitesimals X and Y and the variables s and r are written, according to the operator Γ, as

$$\Gamma\ r = 0 \quad \text{and} \quad \Gamma\ s = 1 \ .$$

The infinitesimal Operator

The equation $\frac{dy}{dx} - F(x, y) = 0$ depends on the variables x and y, however, you can see a y' as another variable and with this the **extended operator** is defined as

$$\Gamma^{(1)} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Y_{[x]}\frac{\partial}{\partial y'}. \tag{11}$$

If we make $\Delta = \frac{dy}{dx} - F(x, y) = 0$ then we can conclude that $\Gamma^{(1)}\Delta = 0$ if and only if $Y_{[x]} = XF_x + YF_y$. It will be written

$$\Gamma^{(1)}\Delta \Big|_{\Delta=0} = 0 \, . \tag{12}$$

Equation (12) leads us to a system of equations whose solution are infinitesimals X and Y, and with them, we can obtain the transformations that reduce equation $\Delta = \frac{dy}{dx} - F(x, y) = 0$ to separable variables.

Extension to higher orders

The n-th largest extension of Γ is given by

$$\Gamma^{(n)} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Y_{[x]}\frac{\partial}{\partial y'} + Y_{[xx]}\frac{\partial}{\partial y''} + \ldots + Y_{[nx]}\frac{\partial}{\partial y^{(n)}} \ ,$$

where each coefficient is given by

$$Y_{[nx]} = D_x\left(Y_{[(n-1)x]}\right) - y^{(n)}D_x(X),$$

and D_x is the differential operator $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + \ldots$.
The invariance condition of the differential equation $\Delta\left(x, y, \ldots, y^{(n)}\right) = 0$ is written as

$$\Gamma^{(n)}\Delta \ \Big|_{\Delta=0} = 0 \ .$$

Example 2: Heat Equation

Let us consider the one dimensional heat equation

$$u_t = u_{xx} \ , \tag{13}$$

where $\Delta = u_t - u_{xx}$. This equation depends of three variables t, x and u, although u_t and u_{xx} also act as variables. Hence, its infinitesimal operators are

$$\Gamma\Delta = \left[T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}\right](u_t - u_{xx}) = 0 \ ,$$

$$\Gamma^{(1)}\Delta = \Gamma\Delta + \left[U_{[t]}\frac{\partial}{\partial u_t} + U_{[x]}\frac{\partial}{\partial u_x}\right](u_t - u_{xx}) = U_{[t]} \ ,$$

$$\Gamma^{(2)}\Delta = \Gamma^{(1)}\Delta + \left[U_{[xx]}\frac{\partial}{\partial u_{xx}} + U_{[xt]}\frac{\partial}{\partial u_{xt}} + U_{[tt]}\frac{\partial}{\partial u_{tt}}\right](\Delta) = U_{[t]} - U_{[xx]} \ .$$

Then equation (14) results

$$U_{[t]} - U_{[xx]} = 0 \quad if \quad \Delta = u_t - u_{xx} = 0. \tag{14}$$

In general, the extended transformations have the form

$$
\begin{aligned}
U_{[t]} &= D_t(U) - u_t D_t(T) - u_x D_t(X)\ , \\
U_{[x]} &= D_x(U) - u_t D_x(T) - u_x D_x(X)\ , \\
U_{[xx]} &= D_x(U_{[x]}) - u_{tx} D_x(T) - u_{xx} D_x(X)\ ,
\end{aligned}
$$

where $D_t(U) = U_t + u_t U_u$ and $D_x(U) = U_x + u_x U_u$ are the total differentials (this is same for X and T). According to these extended transformations, equation (14) is rewritten as

$$
\begin{aligned}
&[U_t - U_{xx}] + [2X_x + T_{xx} - T_t]\, u_t + [X_{xx} - X_t - 2U_{ux}]\, u_x + \\
&\quad [2X_u + 2T_{xu}]\, u_t u_x + [2X_{ux} - U_{uu}]\, u_x^2 + T_{uu} u_t u_x^2 + X_{uu} u_x^3 + \\
&\qquad 2T_x u_{tx} + 2T_u u_x u_{xx} = 0. \qquad (15)
\end{aligned}
$$

Based on this equation, the system of equations results

$$
\begin{array}{llllll}
U_t - U_{xx} = 0 & (1) & 2X_u + 2T_{xu} = 0 & (4) & X_{uu} = 0 & (7) \\
2X_x + T_{xx} - T_t = 0 & (2) & 2X_{ux} - U_{uu} = 0 & (5) & 2T_x = 0 & (8) \\
X_{xx} - X_t - 2U_{ux} = 0 & (3) & T_{uu} = 0 & (6) & 2T_u = 0 & (9)
\end{array}
$$

The solution of this system of equations has the form

$$
\begin{aligned}
T(t) &= c_0 + 2c_1 t + 4c_2 t^2 \ , \\
X(t,x) &= c_3 + 2c_4 t + c_1 x + 4c_2 tx \ , \\
U(t,x,u) &= \left(c_5 - 2c_2 t - c_4 x - c_2 x^2\right) u + C(t,x) \ .
\end{aligned} \tag{16}
$$

where $C(t,x)$ is a function that satisfies the heat equation $C_t = C_{xx}$ and each c_i is constant.

Exact solutions via symmetry analysis through invariance conditions have the form

$$Tu_t + Xu_x = U$$

where T, X and U are the infinitesimals obtained in (16). The solutions to the heat equation are achieved by giving value to each constant, one for one of them and zero for the others. Let's see

i. $c_1 = 1$, $c_i = 0$ for the rest of the indexes and $C(t, x) = 0$. The invariance condition is written as

$$2tu_t + xu_x = 0 \, .$$

By characteristics method, the solution $u = F\left(\frac{x}{\sqrt{t}}\right)$ results. As u satisfies the heat equation (13) then it is shown that

$$u(t, x) = k_1 \; erf\left(\frac{x}{\sqrt{t}}\right) + k_2 \quad \text{where} \quad erf(w) = \int_{-\infty}^{w} e^{-\frac{r^2}{4}} dr \, .$$

Exact solutions in the heat equation

ii. $c_2 = 1$, $c_i = 0$ for the remaining indexes and $C(t,x) = 0$. The invariance condition and the exact solution are

$$\begin{aligned} 4t^2 u_t + 2txu_x &= -(2t + x^2)u \ , \\ u(t,x) &= k_1 \frac{x}{t\sqrt{t}} e^{-\frac{x^2}{4t}} + k_2 \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \ . \end{aligned} \tag{17}$$

iii. $c_3 = 1$, $c_i = 0$ for the rest of the indexes and $C(t,x) = 0$. For this indexes set, the equation is deduced to $dx = 0$, $du = 0$ and $dt = 0$. Whose solutions are constant for each variable. This also happens when $c_5 = 1$.

Obtaining the constants k_1, k_2 depends on the initial or boundary conditions attached to the heat equation.

Heat Kernel

iv. $c_4 = 1$, $c_i = 0$ for the rest of the indexes and $C(t,x) = 0$. The invariance condition leads to the equation $2tu_t = -xu$ whose solution is
$$u(t,x) = \frac{k_1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \ .$$
This solution is known in the literature as **Heat Kernel**, where the constant k_1 is $k_1 = \frac{1}{\sqrt{4\pi}}$.

The solution set of the heat equation (13) is
$$S = \left\{ k_1 \ erf\left(\frac{x}{\sqrt{t}}\right) + k_2, k_3 \frac{x}{t\sqrt{t}} e^{-\frac{x^2}{4t}}, k_4 \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \right\} \ .$$

Gaussian Family

Let X be a random variable with Gaussian distribution of parameters (μ, σ) where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. Its probability density has the form

$$p\left(x, (\mu, \sigma)\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) .$$

Let $\mathcal{M}$ be induced by this family of probability distributions. The Fisher metric g^F (and its inverse) with diagonal representation referring to the parameter set (μ, σ) is

$$g^F = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix} \quad \text{and} \quad g^F = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \frac{\sigma^2}{2} \end{bmatrix} . \tag{18}$$

Operator Laplace Beltrami in the manifold $\mathcal{M}$

The Laplace-Beltrami operator is defined as

$$\Delta_{g^F} u = \frac{1}{\sqrt{\det\left(g^F\right)}} \left[\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{n} \left(g^F\right)^{ij} \sqrt{\det\left(g^F\right)} \frac{\partial u}{\partial x_i} \right) \right] \tag{19}$$

where $\left(g^F\right)^{ij}$ are the components of the inverse and its determinant is $\det\left(g^F\right) = \frac{2}{\sigma^4}$. For a function u defined on the manifold $\mathcal{M}$, its Laplacian is given by

$$\Delta_{g^F} u = \sigma^2 \frac{\partial^2 u}{\partial \mu^2} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial \sigma^2} = \sigma^2 u_{\mu\mu} + \frac{\sigma^2}{2} u_{\sigma\sigma}. \tag{20}$$

By convention, we are going to change the variables μ and σ by x and y, then the Laplacian will be written as

$$\Delta_{g^F} = y^2 u_{xx} + \frac{y^2}{2} u_{yy} \ . \tag{21}$$

On the manifold $\mathcal{M}$, the heat equation has the form $\Delta_{g^F} u = u_t$ that leads to the equation

$$2y^2 u_{xx} + y^2 u_{yy} = 2u_t \ . \tag{22}$$

Applying the previously studied extensions to this heat equation, results

$$2y^2 U_{[xx]} + y^2 U_{[yy]} - 2U_{[t]} + (4yu_{xx} + 2yu_{yy})\, Y = 0 \ . \tag{23}$$

Finding the total differentials and the extended transformations of order 1 and 2, we obtain a system of 31 equations that can be reduced to 17 of them.

System of equations

$$
\begin{array}{llll}
(1) & 2y^2U_{xx} + y^2U_{yy} - 2U_t = 0 & (10) & 2Y_x + X_y = 0 \\
(2) & 2T_t - 2U_u - 2y^2T_{xx} - y^2T_{yy} = 0 & (11) & 4y^2T_x = 0 \\
(3) & 4y^2U_{ux} - y^2\left[2X_{xx} + X_{yy}\right] + 2X_t = 0 & (12) & 2y^2T_y = 0 \\
(4) & 2y^2U_{uy} - y^2\left[2Y_{xx} + Y_{yy}\right] + 2Y_t = 0 & (13) & 2T_u = 0 \\
(5) & 2y^2U_u - 4y^2X_x + 4yY = 0 & (14) & y^2X_u = 0 \\
(6) & y^2U_u - 2y^2Y_y + 2yY = 0 & (15) & 3y^2Y_u = 0 \\
(7) & U_{uu} - 2X_{ux} = 0 & (16) & X_u - 2y^2T_{ux} = 0 \\
(8) & U_{uu} - 2Y_{uy} = 0 & (17) & Y_u - y^2T_{uy} = 0 \\
(9) & X_{uy} + 2Y_{ux} = 0 & &
\end{array}
$$

Whose solution leads to the infinitesimals

$$
T = a_0 + a_1 t\,, \qquad X = A(x,y)\,,
$$
$$
Y = B(x,y)\,, \qquad U = a_1 u + C(x,y,t)\,,
$$

where a_0 and a_1 are constants and C is a function that satisfies the heat equation $2y^2u_{xx} + y^2u_{yy} = 2u_t$.

To obtain some solutions, it will be assumed that the infinitesimals X and Y are polynomials of degree three in the variables x and y, as

$$X = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3 ,$$
$$Y = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3 .$$

Since these infinitesimals satisfy equations $Y_y - X_x = 0$ and $2Y_x + X_y = 0$ then conditions for the coefficients b_i and c_i are found and therefore the solution is written as

$$\begin{aligned} T &= a_0 + a_1 t \\ X &= b_0 + c_2 x - 2c_1 y + \frac{c_4}{2} x^2 - 4c_3 xy - c_4 y^2 + \frac{c_7}{3} x^3 + c_8 x^2 y - 2c_7 xy^2 - \frac{2}{3} c_8 y^3 \\ Y &= c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3 \\ U &= a_1 u + C(x, y, t) . \end{aligned} \tag{24}$$

If we give values to the constant, we can achieve the operators according to the Lie invariance condition.

Since the solutions depend on 13 constants, then the generators of symmetry result when each constant is 1 and the others are zeros and it is assumed that $C(x, y, t) = 0$.

b_0	$\Gamma_1 = \frac{\partial}{\partial x}$		
c_1	$\Gamma_2 = -2y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$	c_8	$\Gamma_9 = \left(x^2y - \frac{2}{3}y^3\right)\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y}$
c_2	$\Gamma_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	c_9	$\Gamma_{10} = y^3\frac{\partial}{\partial y}$
c_3	$\Gamma_4 = -4xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$	c_0	$\Gamma_{11} = \frac{\partial}{\partial y}$
c_4	$\Gamma_5 = \left(-\frac{1}{2}x^2 - y^2\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	a_0	$\Gamma_{12} = \frac{\partial}{\partial t}$
c_5	$\Gamma_6 = y^2\frac{\partial}{\partial y}$	a_1	$\Gamma_{13} = t\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}$
c_6	$\Gamma_7 = x^3\frac{\partial}{\partial y}$		
c_7	$\Gamma_8 = \left(\frac{1}{3}x^3 - 2xy^2\right)\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y}$		

Table 3: Gererators of symmetries for the heat equation defined on $\mathcal{M}$

Exact solutions to the heat equation

Some of the solutions by characteristics method are

Constants	Solutions
$c_1 = 1$	$u = k_1\left(y^2 - \frac{x^2}{2}\right) + k_2$
$c_2 = 1$	$u = k_1 xy + k_2$
$c_{12} = 1$	$u = \frac{k_1}{t}\sin\left(\frac{x}{\sqrt{2ty}}\right) + \frac{k_2}{t}\cos\left(\frac{x}{\sqrt{2ty}}\right)$ for $t > 0$
$b_0 = 1$ $c_5 = 1$ $c_6 = 1$ $c_9 = 1$ $c_0 = 1$ $a_0 = 1$	Each variable is constant

Table 4: Some solutions of the heat equation on $\mathcal{M}$

Future works

1. We should consider the infinitesimals $X(x, y)$ y $Y(x, y)$ as polynomials of degree n and m and find conditions for the exponents and coefficients.
2. Program these processes in MATLAB or Python to be applied in high dimensions.
3. On the manifold induced by the family of q-Gaussian distributions of parameters (μ, σ), give solution to the heat equation via symmetry analysis, where q is the entropy index of Tsallis.

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Thank you!!